The Nature and Nurture of Rectangles

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March 5, 2019

Abstract

An analogy between rectangles and the combination of nature and nurture is a staple of popular discussions of the subject, and has even crossed over into more technical presentations. It is supposed to make the point that nature and nurture are inseparable and cannot be compared (here the details vary). The analogy is based on a misunderstanding, which can be clarified using elementary mathematics. Nature and nurture, like the sides of rectangles, can and have been disentangled. To say that they are *equally essential* is not to say that they are *equally important*.

In 1958, the psychologist Donald Olding Hebb (1904-1985) introduced an analogy between the nature-nurture controversy and areas of rectangles. It has proved highly influential, particularly among those who argue that nature and nurture are logically inseparable, and have expressed the wish that the whole question be dropped. Hebb's textbook original was couched in terms of the area of a farmer's field.[1]

> Sometimes it is recognized that heredity and environment both affect intelligence, but the writer then goes on to say how important each is. The student may find it said for example that 80 per cent of intelligence is determined by heredity, 20 per cent by environment. This statement is, on the face of it, nonsense. It

[1] Hebb 1958, 128-129.

> means that a man would have 80 per cent of the problem-solving ability he would otherwise have had, if he were never given the opportunity to learn a language, never learned how people behave, and so forth. Conversely, it means that 20 per cent of a man's problem-solving capacity will result from a good environment, no matter that heredity is involved, which we know of course is not true. What we must say is that both these variables are of 100 per cent importance: their relation is not additive but multiplicative. To ask how much heredity contributes to intelligence is like asking how much the width of a field contributes to its area.

There are far too many repetitions of this in the literature to enumerate, and we will not dwell on sources, such as general psychology textbooks, whose overall reputation for inaccuracy is notorious.[2] Searching the internet turns up numberless variants. More importantly, the analogy even pops up in the behaviour genetic literature itself. Thus in McLearn, DeFries and Plomin's popular text *Behavior Genetics*, which has gone through many editions since 1973 and a roster of co-authors, we learn that 'It is nonsensical to ask about the separate contributions of length and width to the area of a single rectangle because area is the product of length and width. Area does not exist without both length and width.'[3] The same argument was made by David Lykken. 'It is meaningless to ask whether Isaac Newton's genius was due more to his genes or his environment, as meaningless as asking whether the area of a rectangle is due more to its length or its width'.[4] James Kalat, in *Biological Psychology* (1984) is emphatic. 'As an analogy, consider the question "What is more important for the area of a rectangle, the height or the width?". Obviously both are equally important.'[5] In the contest between nature and nurture, all must have prizes!

Yet nature and nurture, more specifically non-shared and shared environment, have well-estimated contributions at a population level.

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[2] For some examples see Gray 2002, 377; Lerner 1989, 89; Erlich 2000, 334. See also Sesardic 2006 for some more citations.

[3] Plomin et al. 2012, 89.

[4] Lykken 1998.

[5] Kalat 1984.

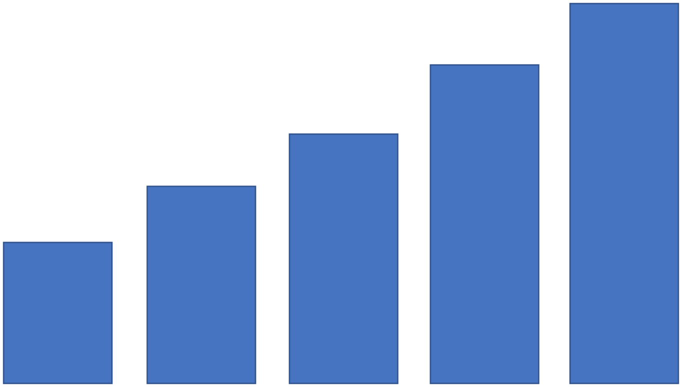

Figure 1: *A population of rectangles in which differences in height matter more for area than differences in width.*

They are not equal. They are not hopelessly entangled. Nature dominates shared environment, which is nearly negligible beyond adolescence. And this is not alchemy.[6] Heritability is a subject that is particularly prone to misinterpretation and misrepresentation. The philosopher of science Neven Sesardic has made a concerted attempt to work through this thicket in his *Meaning of Heritability* (2006), but could only cover the major points.[7] A whole encyclopedia could be compiled on the topic. Like most well-informed commenters, Sesardic deals with the rectangle argument from the point of view of *populations*.

Given a population or collection of rectangles, we can readily establish which side contributes more to the areas, or whether there is little difference. That depends on the population in question. It is a simple matter to give examples of collections of rectangles where differences in length make more difference to areas than differences in breadth, as one rectangle succeeds another. In Figure 1 we simply keep breadth constant, while increasing height. Since there are no differences in width, they do not explain any differences in area at all.

That nicely illustrates the point that in behaviour genetic analyses,

[6] Plomin 2018; Polderman et al. 2015.
[7] Sesardic 2006.

we are concerned with differences, and the way that the world happens to be, rather than the way it might be. However, there is a much simpler way of dealing with the rectangle analogy, which is misconceived even in the case of *one single rectangle.* Doing so reveals the astonishing extent of innumeracy among those who are commonly trusted as sources of fact. In this exercise, the more numerate reader will surely lose patience. Sometimes swatting a gnat requires a sledgehammer. I can only plead that somebody had to do it.

To fix ideas, let us discuss a rectangle R, which has sides a and b. As any school-child knows, it has area $A = a \times b$. It is certainly true that if $a = 0$ or $b = 0$ then $A = 0$, which is usually what we mean when we say that a rectangle requires both width and breadth, if it is to have an area at all. Admittedly those given to abstraction may prefer to think of a line segment (one side) as a degenerate rectangle with one side of length zero and and one non-zero, The degenerate rectangle would occupy zero area, as lines do. But most people do not think that way. Well and good, we will require that both sides be non-zero in this discussion, so that the rectangle has non-zero area.

In general, it is correct to say that, given a rectangle of unspecified dimensions, we cannot say which side contributes more to its area. But that is only because *the dimensions have not been specified or constrained in any way.* Once they are, we may do exactly that: determine which contributes *more* to its area, or perhaps that they do so *equally.* Let's take a practical example, the rectangle with sides of length 3 and 4. Verily it has area $3 \times 4 = 12$. To figure out which side contributes more we take proportions and compare them, an age-old idea. We see that

$$\frac{3}{12} < \frac{4}{12} \iff 3 < 4 \tag{1}$$

which is true. That was easy. Proportionally, 4 contributes more to the area than 3, because $4 > 3$. This is true even though both sides are greater than zero. Of course, if both sides are, say, 3, then they contribute equally to the area, 9. More generally, for any sides a and b where $a > 0$ and $b > 0$ (which is to say that the rectangle has a non-zero area)

$$\frac{a}{a \times b} \leq \frac{b}{a \times b} \iff a \leq b, \tag{2}$$

It is flatly false that we cannot disentangle the contributions to the area of a rectangle merely because both sides must be non-zero. Stating the formula $A = a \times b$ disentangles their contribution to the area at once. If we had no formula we might not disentangle them. But perhaps the reader prefers addition to multiplication, a feeling which appears to have motivated the references to *products*, quoted above, as if they were something special.

Consider a large man, say Hercules, who is standing on a scale. A bee has perched on his nose. The weight of this ensemble, Hercules and bee, is found by addition (the reader was warned). $W = h + b$ where h is the weight of Hercules, and b is the weight of the bee. Now it is true that in relative terms, the weight of the bee contributes little, but it is certainly non-zero. Supposing that we want the exact weight of the ensemble, it makes no sense to speak of that weight without taking into account the weight of the bee, or else we have only a close approximation. Pedants may go so far as to say that it is *nonsensical* to speak of the weight of the ensemble without taking both h and b into account. Yet even though $W = h + b$ with $h > 0$ and $b > 0$, we may still *compare* the contributions of h and b. Hercules contributes vastly more to the weight of the ensemble than the bee, which is to say that $h \gg b$, where '$\gg$' means 'is much larger than'. So much so that the sane would simply ignore the bee.

Though these two problems may appear to be distinct, one involving multiplication and the other addition, they are exactly the same. We are dealing with positive numbers, and logarithms map one space to the other, transforming multiplication into addition: $\log(a \times b) = \log(a) + \log(b)$. To get back, use the inverse operation $\exp(\log x) = x$. The wonderful thing about log and exp is that they are order-preserving (monotonic), so that

$$\begin{aligned} \log(a) \leq \log(b) &\iff a \leq b; a, b > 0 \\ \exp(x) \leq \exp(y) &\iff x \leq y \end{aligned} \tag{3}$$

So, if one prefers, the area of a rectangle can be written as

$$\log(A) = \log(a \times b) = \log(a) + \log(b) \tag{4}$$

and the problem 'does a contribute more to A than b?' is really the same problem as 'does x contribute more to $z = x + y$ than y?' Precisely the problem of Hercules and the bee. If you can solve one then you can solve the other. And you can solve both. Which is why we know that the total volume enclosed by the Alaska oil pipeline is due rather more to its prodigious length than it is to its more modest diameter.

The crux of the fallacious analogy lies in the subtle confusion between the requirement that a rectangle have both length and breadth (they are 'essential') and the idea that they are equally 'important'. To be *equally essential* is not necessarily to be *equally important*.

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